Analytical study of coupled two-state stochastic resonators

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The two-state model of stochastic resonance is extended to a chain of coupled two-state elements governed by the dynamics of Glauber's stochastic Ising model. Appropriate assumptions on the model parameters turn the chain into a prototype system of coupled stochastic resonators. In a weak-signal limit, analytical expressions are derived for the spectral power amplification and the signal-to-noise ratio of a two-state element embedded into the chain. The effect of the coupling between the elements on both quantities is analyzed and array-enhanced stochastic resonance is established for pure as well as noisy periodic signals. The coupling-induced improvement of the signal-to-noise ratio compared to an uncoupled element is shown to be limited by a factor 4, which is only reached for vanishing input noise. [S1063-651X(98)02909-2]

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I. INTRODUCTION

The essential point of stochastic resonance [1] is the following: If a stochastic resonator is subjected to an external influence, commonly referred to as the signal, its response exhibits signal features, which are most pronounced at a certain level of noise present in the system. The fascinating aspect of stochastic resonance is therefore that, counterintuitively, an increasing noise level does not steadily deteriorate the transmission of the signal through the resonator. On the contrary, noise may rather be used to optimize the transmission process [2].

The implications of these effects are intensively investigated. In biological systems, for example, stochastic resonance apparently plays a role in the neural transmission of information [3]. From a technical point of view one might possibly see the emergence of a novel type of detector that incorporates an optimal amount of noise to perform best. In this regard superconducting quantum interference devices (SQUID's) have been studied [4]. They may be made to detect weak magnetic fields reasonably well without the usual costly shielding from environmental noise.

In recent years it was shown that the performance of a single stochastic resonator can be enhanced, if it is embedded into an ensemble of other stochastic resonators that are properly coupled [5–9]: Compared to being operated isolated, the response of the resonator to the signal increases within the coupled ensemble. However, if the coupling becomes too strong this response is found to deteriorate again. Thus in an ensemble of stochastic resonators the coupling strength turns out to be a second design parameter: Apart from the noise level, it can be tuned to achieve an optimal performance of the embedded resonator. This effect was called *array-enhanced stochastic resonance* [8]. It will possibly find technical exploitation and might also be relevant to biological systems, for example, coupled neurons.

The aim of the present paper is a further investigation of the phenomena related to array-enhanced stochastic resonance. To this end we study a simple prototype system of coupled two-state stochastic resonators under periodic modulation. The model we propose allows us to analytically calculate the weak-signal limits of two prominent stochastic-resonance quantities, spectral power amplification (SPA) and signal-to-noise ratio (SNR), respectively. Following the concept of array-enhanced stochastic resonance, both quantities refer to the response of a resonator as part of the coupled ensemble. The impact of the coupling on both response measures will be studied.

There is a close link of the present analysis to the work presented in [5] and [8], which to our knowledge is the most comprehensive in the field. However, compared to [5], where the SPA of a system of globally coupled bistable elements was studied analytically, too, the present approach offers a more refined description, since it allows us to avoid a mean-field approximation. Compared to [8], on the other hand, where the SNR of a chain of locally coupled elements was investigated in a (rather extensive) simulation, the SNR can now be obtained analytically. For a quick review of the results the reader is referred to the third and fourth passage of the summary section.

From a conceptual point of view the model proposed here can be seen as an extension of the two-state model developed by McNamara and Wiesenfeld [10] to study stochastic resonance in noisy bistable systems. Instead of considering individual two-state elements as in [10], these elements are now arranged in a chain. A simple next-neighbor interaction is introduced that brings the model close to the system studied in [8]. The interaction is chosen in such a way that the resulting evolution of the elements is given by Glauber's stochastic Ising model [11]. A detailed description of this approach is given in the next section, which also provides the necessary background. We note that the connection to the Glauber model is made for mathematical convenience. We are not concerned with the observation of stochastic resonance in Ising systems, which was the central theme in [12] and [13]. Our purpose will also lead to the assumption of Arrhenius-type transition rates of uncoupled unmodulated elements, which is unusual in the context of Ising systems. The first results of the present analysis were published in [14].

A further interesting approach to arrays of stochastic resonators is centered around a response of a more collective nature. Here stochastic resonance is studied in the summed output of N resonators, which need not necessarily be coupled [15–17]. Recently, it was shown analytically that the SNR of this output approaches the input SNR, if a sufficiently large number of uncoupled resonators is used [18]. Whether in some way coupling may still be beneficial for the collective response of the present model will be the subject of a future study. Only the summed output of a very large number of resonators will be briefly discussed here. It will be shown that in this case the SNR always deteriorates under coupling. This indicates that the coupling-induced improvement of the performance of stochastic resonators associated with array-enhanced stochastic resonance is a local rather than a global effect.

II. FROM SINGLE TO COUPLED TWO-STATE STOCHASTIC RESONATORS

An early theory of stochastic resonance in noisy bistable systems with one (generally continuous) variable was worked out by McNamara and Wiesenfeld [10]. They studied the effect of a periodic modulation of these systems in a way that is independent of the precise dynamics involved. To this end the behavior of the bistable system was approximated by a random telegraph process. For simplicity this process was taken to be symmetric, randomly switching between two states $\sigma=\pm c$. For convenience we shall consider here $c=1$. The probabilities $p(\sigma)$ to find the process in state σ satisfy $p(\sigma)+p(-\sigma)=1$ and their time-evolution is governed by

$$\dot{p}(\sigma)=-\dot{p}(-\sigma)=W(-\sigma)p(-\sigma)-W(\sigma)p(\sigma), \quad (1)$$

where $W(\sigma)$ denotes the rate of the transition $\sigma\rightarrow-\sigma$. These rates must be extracted from the precise dynamics at hand. To build a theory of stochastic resonance they were assumed to be approximately given by [10]

$$W(\sigma)=\frac{\alpha}{2}[1-\sigma\delta\cos(\omega t+\phi)] \quad (2)$$

with δ being the small parameter of the theory. Obviously, at this level the periodic modulation of the bistable system with (angular) frequency ω and phase ϕ is taken into account by the cosine term only.

This simple model allows us to calculate the long-time limit of the power spectrum of the random telegraph process averaged over a uniformly distributed phase ϕ. The spectrum consists of a continuous part and a δ function at modulation frequency ω. The continuous part of the spectrum is the Lorentzian of the unperturbed process times a frequency-dependent prefactor. The latter is close to 1 and governs the modulation-induced transfer of broadband power to the δ peak in the spectrum.

From this spectrum McNamara and Wiesenfeld obtained their central theoretical result with respect to stochastic resonance: An exact analytical expression for the signal-to-noise ratio (SNR) of the two-state process. This SNR is defined as the ratio of the weight of the δ function to the continuous part of the spectrum at modulation frequency. In view of the present work we will neglect the signal-induced suppression of the continuous part of the spectrum and consider the linear-response approximation of the McNamara-Wiesenfeld result with respect to the small parameter δ. Doing so, one finds for the SNR

$$R_0=\frac{\pi}{4}\alpha\delta^2. \quad (3)$$

If the approximation (2) is performed on a particular noisy bistable system subject to a periodic modulation, the parameters α and δ become functions of noise strength and modulation amplitude, respectively. The dependence of the SNR on the noise intensity can then be studied and the occurrence of stochastic resonance may be established for that particular system.

As an example, the overdamped double-well system was given in [10]. The model equation reads

$$\dot{x}=x-x^3+A\cos(\omega t+\phi)+\sqrt{2D}\xi(t), \quad (4)$$

where $\xi(t)$ is Gaussian white noise with $\langle\xi(t)\rangle=0$ and $\langle\xi(t)\xi(t+\tau)\rangle=\delta(\tau)$. Using a modified Kramers formula for the transition rates valid for sufficiently low modulation frequencies, McNamara and Wiesenfeld found

$$\alpha=\frac{\sqrt{2}}{\pi}\exp\left(-\frac{1}{4D}\right), \quad \delta=\frac{A}{D}. \quad (5)$$

The central idea of the present work is to pass from the single bistable element to a set of coupled bistable elements by extending the two-state model of McNamara and Wiesenfeld. To this end we consider a chain of two-state elements that for convenience is taken to be of infinite length. If the elements interact, the simple gain-loss balance (1) has to be modified to describe the evolution of the set of probabilities $p(\overline{\sigma},t)$ to find the chain in a particular configuration $\overline{\sigma}=(\ldots,\sigma_{k-1},\sigma_k,\sigma_{k+1},\ldots)$ at time t. Introducing a formal operator F_k defined for any function $f(\sigma_k)$ by $F_k f(\sigma_k)=f(-\sigma_k)$, the new gain-loss balance reads

$$\dot{p}(\overline{\sigma})=\sum_k (F_k-1)W_k(\sigma_k)p(\overline{\sigma}). \quad (6)$$

With the initial condition $p(\overline{\sigma},t_0)=\delta_{\overline{\sigma}\overline{\sigma_0}}$, the $p(\overline{\sigma},t)$ are interpreted as the transition probabilities $p(\overline{\sigma},t|\overline{\sigma}_0,t_0)$ of a Markovian process with infinitely many discrete components. The statistics of the process is fully determined by Eq. (6).

To introduce interactions we assume that the transition rates (2) of an element depend on the states of its next neighbors. A simple choice for this coupling controlled by a parameter γ is

$$W_i(\sigma_i)=W(\sigma_i)\left[1-\frac{\gamma}{2}(\sigma_{i-1}+\sigma_{i+1})\sigma_i\right], \quad (7)$$

where $W(\sigma_i)$ is the McNamara-Wiesenfeld rate (2). With positive γ, neighboring elements prefer to be in the same state, whereas they tend to be in opposite states if γ is negative. Both tendencies grow with growing coupling strength $|\gamma|$. To avoid negative transition rates $|\gamma|\leqslant 1$ has to be required. We note that the assumed type of coupling is thus able to model ferromagnetic-type interactions for which a coupling-induced improvement of the performance of stochastic resonators was found in [5] and [8]. For positive γ, the present model is particularly close to the model simulated in [8], since both are linear arrays of next-neighbor-coupled elements. For convenience we shall often use the terms ferromagnetic or antiferromagnetic coupling instead of coupling with positive or negative γ.

Of course, other types of transition rates (7) could be considered as well. The advantage of the present choice is that most of the relevant stochastic properties of the resulting model are already known. They were studied by Glauber, who introduced this model as a stochastic form of the Ising model [11]. [We note that in the Glauber model the term $\delta \cos(\omega t+\phi)$ in Eq. (7) is replaced by a general time-dependent parameter β.] As in the McNamara-Wiesenfeld model, it is again possible to find an analytical expression for the SNR in leading order of the modulation parameter δ, which is the subject of the next section. The respective analysis is essentially an exploitation of Glauber's work.

It may be possible to reduce the dynamics of some coupled systems to the two-state model given by Eqs. (6) and (7) and hence to directly employ the SNR formula derived in the next section. However, this approach is certainly more involved than the rate expansion (2) needed to make use of the McNamara-Wiesenfeld SNR. We shall not pursue this more general aspect of the model in further detail. Instead we are interested in devising a simple prototype system that allows us to study the impact of the coupling on the stochastic-resonance effect. To this end we are looking for simple assumptions on the dependence of the model parameter α, γ, and δ on some noise intensity and signal amplitude, respectively. We proceed in two steps.

First, we retain Glauber's original relation of the present model to the Ising model given by the Hamiltonian

$$\mathcal{H}=-J\sum_k \sigma_k\sigma_{k+1}-\mu H\sum_k \sigma_k . \tag{8}$$

This relation is defined by an assumption on the probability $p(\sigma_k)$ to find the kth element in state σ_k, if all other elements are fixed: In an adiabatic limit the $p(\sigma_k)$ of both models are to be identical. This implies that

$$\frac{p(\sigma_k)}{p(-\sigma_k)}=\frac{W_k(-\sigma_k)}{W_k(\sigma_k)}=\exp\left(-\frac{\mathcal{H}(\sigma_k)-\mathcal{H}(-\sigma_k)}{kT}\right) \tag{9}$$

holds, which allows us to establish a relation between the parameters of both models. One finds $\gamma=\tanh(2J/kT)$ and $\delta \cos(\omega t+\phi)=\tanh(\mu H/kT)$, respectively [11]. The role of the noise intensity is thus played by the temperature T. To skip unnecessary parameters we set $k=1$ and $\mu=1$. In the present paper we assume that H, which shall be referred to as the signal, is given by

$$H=H_0 \cos(\omega t+\phi). \tag{10}$$

For weak amplitudes $H_0 \ll T$ this results in $\delta=H_0/T$. [In Sec. V we shall also consider periodic signals (10) with an additional noisy background.]

Second, a plausible assumption has to be made on the temperature dependence of the time-scale parameter α, which is not affected by requiring Eq. (9). It is assumed that it retains the qualitative dependence (5) found by McNamara and Wiesenfeld for the double-well system, i.e., $\alpha=\alpha_0 \exp(-1/T)$ with properly scaled temperature T. This dependence is typical for many rate processes [19]. An appropriate temperature dependence of α is also necessary to turn the elements into stochastic resonators for vanishing coupling. For simplicity $\alpha_0=1$ is considered.

The prototype system we are looking for is thus specified by Eqs. (6) and (7) together with

$$\alpha=\exp\left(-\frac{1}{T}\right), \quad \gamma=\tanh\left(\frac{2J}{T}\right), \quad \delta=\frac{H_0}{T}. \tag{11}$$

We emphasize that this system is not an approximation of the Ising system (8) in the same way as the double-well system (4) is approximated by Eqs. (1), (2), and (5), respectively. The devised prototype system appears to capture in a simple way the essence of the effect of coupling and forcing on coupled continuous-variable systems such as the double well.

To motivate this, let $\Delta U(\sigma)$ denote the height of the barrier to be surmounted in order to escape from state σ. In a very simple approximation the effect of next-neighbor coupling and forcing on the barrier height could be modeled as

$$\begin{gathered}\Delta U(\sigma_k)=\Delta U_0+\Delta U_{\text{add}}(\sigma_k),\\ \Delta U_{\text{add}}(\sigma_k)=J\sigma_k(\sigma_{k+1}+\sigma_{k-1})+H_0\sigma_k \cos(\omega t+\phi).\end{gathered} \tag{12}$$

Comparing this to Eq. (8) one finds that within this simple picture the effect of coupling and forcing on the barrier heights is the same as their effect on the energy levels of the Ising model. (We note that for coupled double-well systems the chosen approximation would only hold for weak coupling. For strong coupling the systems may no longer be bistable and the entire concept of a simple-minded two-state approximation breaks down.)

Within an adiabatic limit and with some noise strength T the transition rate $W(\sigma_k)$ of the barrier system would roughly read

$$W(\sigma_k)=\exp\left(-\frac{\Delta U_0}{T}\right)\exp\left(-\frac{\Delta U_{\text{add}}(\sigma_k)}{T}\right). \tag{13}$$

From here rates of the form (7) together with parameters (11) can be obtained in two steps. First, the factor related to ΔU_0 is kept as the parameter α with a respective choice of ΔU_0. Second, the factor related to ΔU_{add}, which describes the effect of forcing and coupling, is replaced by the respective term of Eq. (7), which is easier to handle. A connection between both types of rates is again made by requiring that they lead to identical stationary distributions in the sense of Eq. (9).

A particular close relation between the Glauber-dynamics model and the barrier system should thus arise within the regime of quasistatic response. Nevertheless, we shall not restrict the following investigation of the simple prototype system to the quasistatic regime. Whether or not the suggested model provides insights that are also relevant to more realistic settings and possibly helpful in the design of coupled stochastic resonators remains to be seen.

III. ANALYTICAL SOLUTIONS

The aim of this section is to calculate the SNR of the response of a two-state element embedded into the chain of coupled elements given by Eqs. (6) and (7). This calculation will be done in terms of the parameters α, γ, and δ and is essentially based on Glauber's work. We shall only take account of terms in leading order of the modulation amplitude δ. The resulting SNR will then be an extension of the simplified McNamara-Wiesenfeld result (3). At the end of the section the general result will be applied to the prototype system specified by Eq. (11). In addition to the SNR we shall also give an analytical expression for the so-called spectral power amplification (SPA).

Following Glauber, we start by calculating the averages $\langle\sigma_k(t)\rangle$. From Eq. (6) one derives $d\langle\sigma_k\rangle/dt = -2\langle\sigma_k W_k(\sigma_k)\rangle$. It results in

$$\frac{d\langle\sigma_k\rangle}{d(\alpha t)} = -\langle\sigma_k\rangle + \frac{\gamma}{2}[\langle\sigma_{k-1}\rangle + \langle\sigma_{k+1}\rangle] + \left[1 - \frac{\gamma}{2}(r_{k-1,k} + r_{k,k+1})\right]\delta\cos(\omega t + \phi) \quad (14)$$

with $r_{i,j}(t) = \langle\sigma_i(t)\sigma_j(t)\rangle$. A closed set of equations for the $\langle\sigma_k(t)\rangle$ is found by linearizing Eq. (14) in δ. In this case the $r_{i,j}$ can be taken from the unperturbed model ($\delta=0$). After long times these $r_{i,j}$ read $r_{i,j} = \eta^{|i-j|}$ with $\eta = \gamma^{-1}(1 - \sqrt{1-\gamma^2})$ [11].

The resulting long-time-limit set of equations can be further simplified. Since all elements are forced uniformly, a particular element cannot be distinguished from any other, once the initial distribution has been forgotten. Hence, all elements will have identical statistics and the indices in Eq. (14) can be skipped. Together with the previous assumption one obtains

$$\frac{d\langle\sigma\rangle}{dt} = -\alpha(1-\gamma)\langle\sigma\rangle + \alpha\sqrt{1-\gamma^2}\,\delta\cos(\omega t + \phi). \quad (15)$$

Obviously, the long-time dynamics of the average $\langle\sigma\rangle$ is identical to the long-time dynamics of the average of an uncoupled element with rescaled relaxation rate $\alpha(1-\gamma)$ and rescaled modulation term $\alpha\sqrt{1-\gamma^2}\delta$. The resulting long-time limit of the averaged state simply reads

$$\langle\sigma(t)\rangle = q\cos(\omega t + \phi + \psi),$$
$$q = q_s\left(1 + \frac{\omega^2}{\alpha^2(1-\gamma)^2}\right)^{-1/2}, \quad \tan\psi = -\frac{\omega}{\alpha(1-\gamma)}, \quad (16)$$

where q_s is the response to static signals ($\omega=0$):

$$q_s = \delta\sqrt{\frac{1+\gamma}{1-\gamma}}. \quad (17)$$

Formally this static response might even exceed $q_s=1$. We conclude that in this case δ is not sufficiently small and the linearization of Eq. (14) is no longer justified. Later on it will be shown that this does not amount to a considerable restriction.

The power spectrum of an element is determined from the average $\langle\sigma_k(t)\sigma_k(t+\tau)\rangle$. As in [10], its t dependence is removed by averaging $\langle\ \rangle_\phi$ over a uniformly distributed initial phase ϕ of the modulation term in Eq. (7). Introducing the correlation function $c_{kk}(t,\tau) = \langle[\sigma_k(t) - \langle\sigma_k(t)\rangle][\sigma_k(t+\tau) - \langle\sigma_k(t+\tau)\rangle]\rangle$ one finds

$$\langle\langle\sigma_k(t)\sigma_k(t+\tau)\rangle\rangle_\phi = \langle c_{kk}(t,\tau)\rangle_\phi + \frac{q^2}{2}\cos(\omega\tau). \quad (18)$$

The second term on the right-hand side contributes a δ function with weight πq^2 at signal frequency ω to the (one-sided) power spectrum as defined below. The first term, on the other hand, forms the continuous part of the spectrum.

For the purpose of calculating the SNR in leading order of the modulation amplitude δ it is sufficient to approximate this continuous part by the power spectrum of the unperturbed model. This implies that we neglect any possible signal-induced transformation of the continuous part of the spectrum as we did to obtain the simplified McNamara-Wiesenfeld result (3). For the unperturbed model the long-time correlation function $c(\tau)$ is not affected by the average $\langle\ \rangle_\phi$. It also no longer depends on the index of the element. According to Glauber, $c(\tau)$ reads

$$c(\tau) = e^{-\alpha|\tau|}\sum_{n=-\infty}^{+\infty}\eta^{|n|}I_n(\alpha\gamma|\tau|) \quad (19)$$

with modified Bessel functions $I_n(\alpha\gamma|\tau|)$. Using the relation [20]

$$\int_0^{+\infty} e^{-ax}I_\nu(bx)dx = \frac{1}{\sqrt{a^2-b^2}}\left(\frac{b}{a+\sqrt{a^2-b^2}}\right)^\nu, \quad (20)$$

which holds for $\mathrm{Re}(\nu) > -1$ and $\mathrm{Re}(a) > |\mathrm{Re}(b)|$, the power spectrum $s(\Omega) = 2\int_0^{+\infty}\cos(\Omega\tau)c(\tau)d\tau$ is calculated. For the one-sided spectrum defined by $S(\Omega) = s(\Omega) + s(-\Omega)$ at $\Omega > 0$ we eventually obtain

$$S(\Omega) = 4\,\mathrm{Re}\left(\frac{1+\eta s_2}{s_1(1-\eta s_2)}\right) \quad (21)$$

with $s_1 = \sqrt{(\alpha + i\Omega)^2 - (\alpha\gamma)^2}$ and $s_2 = \alpha\gamma(\alpha + i\Omega + s_1)^{-1}$.

Now the SNR of the response of a two-state element embedded into the chain can be calculated, which is the central result of this section. It reads

$$R^* = \frac{\pi q^2}{S(\omega)}. \quad (22)$$

For vanishing coupling it reduces to the simplified McNamara-Wiesenfeld result (3).

In addition to the effect of the coupling on the SNR, we also wish to study its impact on the δ peak in the power spectrum. In this regard the spectral power amplification (SPA) is a convenient measure (cf., e.g., Jung in [2]). It is defined as the ratio of power contained in the signal peaks of output to input spectrum, i.e., the ratio of the weights of the respective δ functions. Because of its dependence on the input spectrum, the SPA depends on the precise dynamics to be modeled by the two-state chain.

For the prototype system specified by Eq. (11), the SPA reads $\rho=(q/H_0)^2$, or explicitly

$$\rho=\rho_s\left(1+\frac{\omega^2\exp\left(\frac{2}{T}\right)}{\left[1-\tanh\left(\frac{2J}{T}\right)\right]^2}\right)^{-1}, \tag{23}$$

$$\rho_s=\frac{1}{T^2}\exp\left(\frac{4J}{T}\right), \tag{24}$$

where ρ_s is the SPA of static signals. Obviously, the SPA no longer depends on the signal amplitude H_0. For convenience we also remove the H_0 dependence of the SNR by considering

$$R=\frac{\pi q^2}{H_0^2 S(\omega)}=\frac{\pi\rho}{S(\omega)}. \tag{25}$$

For the prototype system the parameter η involved in the unperturbed spectrum (21) simplifies to $\eta=\tanh(J/T)$ [11].

Both quantities, SPA (23) and rescaled SNR (25), only depend on three parameters: temperature T, coupling strength J, and signal frequency ω. Restrictions arise from $H_0 \ll T$, which lead to a simple expression for δ, as well as from the linearization of Eq. (14), which results in an upper bound on δ as discussed in connection with Eq. (17). Together one finds

$$H_0 \ll \min(T, T\exp(-2J/T)). \tag{26}$$

It implies that finite H_0 places a lower bound on T and an upper bound on J. However, these restrictions are rather weak: H_0 can be made arbitrarily small because its size is immaterial within the present weak-signal approximation.

We are now in a position to study the impact of the coupling on the response measures SPA and SNR of a single resonator embedded into the chain. This will be the subject of the following two sections.

IV. COUPLING AND SPECTRAL POWER AMPLIFICATION

The SPA (23) has a unique maximum over temperature T and coupling parameter J for any time-dependent signal (10). The maximum SPA is obtained for a frequency-dependent value J_{max}, which is always positive. In other words, a properly tuned ferromagnetic-type coupling yields the best SPA performance of the two-state resonator element, which is embedded into the chain.

For our simple model this maximum can be studied analytically. The partial derivatives of the SPA with respect to T and J are found to vanish at pairs (T_{max}, J_{max}) given by

$$J_{max}=-(T_{max}/4)\ln(2T_{max}-1) \tag{27}$$

and

$$\omega^2=\exp\left(-\frac{2}{T_{max}}\right)\frac{(2T_{max}-1)^2}{T_{max}(1-T_{max})}, \tag{28}$$

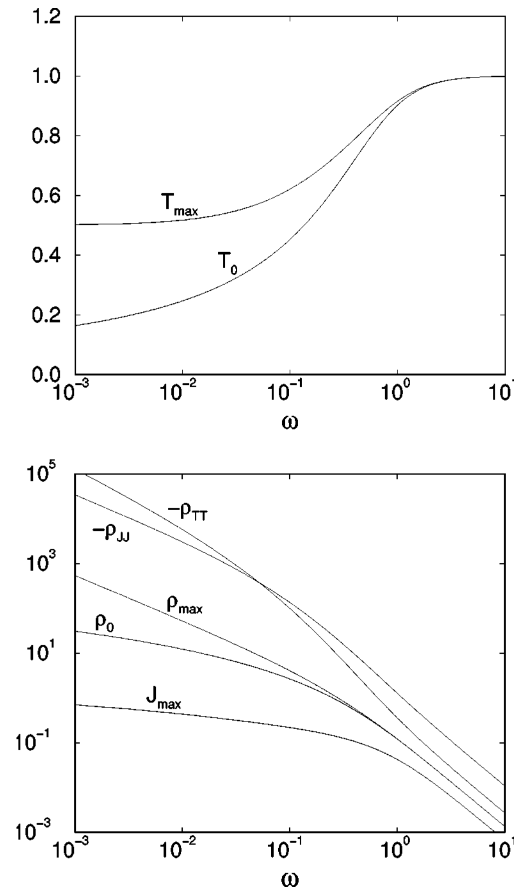

FIG. 1. Features of the unique maximum of the SPA ρ over temperature T and coupling strength J shown in dependence on the signal frequency ω. Plotted are the position (T_{max}, J_{max}) of the maximum, its height ρ_{max}, and the sharpness of the maximum expressed by the curvatures ρ_{TT} and ρ_{JJ}, respectively. In the uncoupled model ($J=0$) the SPA has a maximum at T_0 with a height ρ_0.

respectively. For J_{max} to be real and finite, $T_{max}>1/2$ has to hold. With this restriction, Eq. (28) has a unique solution $1/2<T_{max}<1$ for any given $\omega>0$. The corresponding J_{max} is found via Eq. (27). Within the given range of T_{max} one can easily check on Eq. (27) that J_{max} is indeed always positive. An inspection of the second derivatives of the SPA finally reveals that there is a maximum at (T_{max}, J_{max}). Its peak value is implicitly given by

$$\rho_{max}=\frac{1-T_{max}}{T_{max}^2(2T_{max}-1)}. \tag{29}$$

From Eqs. (27), (28), and (29) the following features of the maximum can be derived, which are shown in Fig. 1: Tuning the signal frequency ω from very large to vanishing small values, the temperature T_{max} falls from 1 to 1/2. At the same time, the coupling strength J_{max} as well as the peak height ρ_{max} increase from vanishing small to very large values. In addition, an analysis of the curvatures ρ_{JJ} and ρ_{TT}, both expressed in terms of T_{max}, shows that the sharpness of the maximum grows as its height increases. This indicates

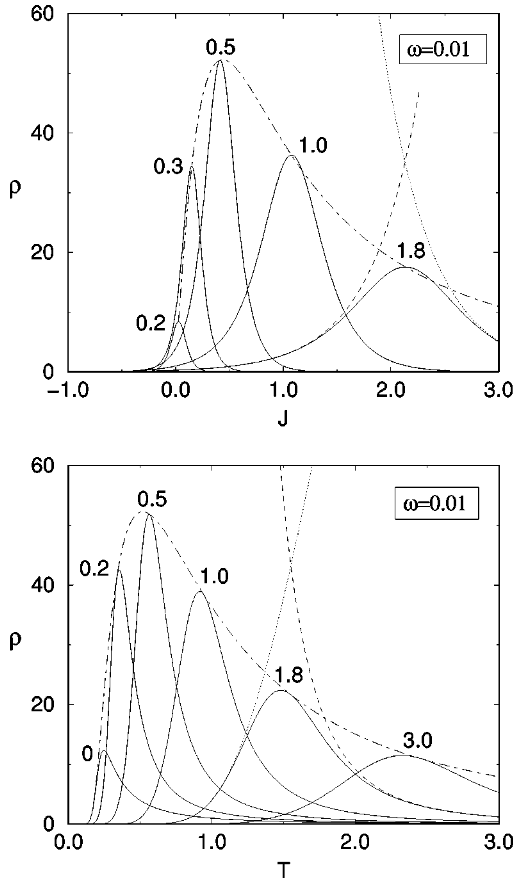

FIG. 2. Typical qualitative behavior of the SPA ρ at low signal frequencies for various values of coupling strength J (top) and temperature T (bottom). Dashed and dotted curves correspond to SPA limits discussed towards the end of this section. The dash-dotted curves represent the SPA at optimal coupling.

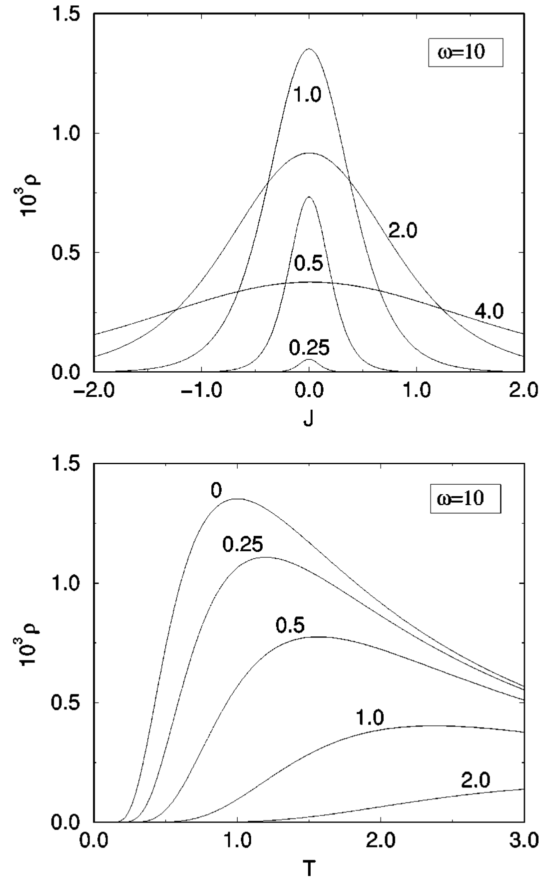

FIG. 3. Typical qualitative behavior of the SPA ρ at high signal frequencies for various values of coupling strength J (top) and temperature T (bottom).

that to achieve optimal performance the system parameters T and J have to be tuned with increasing accuracy as the signal frequency decreases.

Figure 1 also includes a comparison to the SPA of an uncoupled element. At given $\omega > 0$, this SPA has a maximum located at a temperature T_0, implicitly given by

$$\omega^2 = \exp\left(-\frac{2}{T_0}\right)\frac{T_0}{1-T_0} \tag{30}$$

on the interval $0 < T_0 < 1$, whereby its peak value is found to be $\rho_0 = (1-T_0)/T_0^2$. It can be shown from Eqs. (28) and (30) that $T_{\max}$ always exceeds T_0. The fact that $J_{\max}$ was found to be never zero also implies that $\rho_{\max}$ always exceeds ρ_0. However, looking at Fig. 1 it is obvious that at frequencies $\omega > 1$ both compared quantities of the coupled element approach those of the uncoupled one and $J_{\max}$ approaches zero. The coupling-induced increase in the SPA is thus vanishingly small at sufficiently high frequencies of the signal.

This comparison allows us to distinguish between two different types of SPA behavior: At low frequencies the SPA is enhanced under ferromagnetic coupling, whereas at high frequencies it is basically not, although a tiny increase still occurs. Both situations are illustrated at selected frequencies in Fig. 2 and Fig. 3, respectively. The low-frequency SPA exactly reproduces the qualitative effect found analytically by Jung *et al.* [5] in a system of globally interacting elements. The high-frequency behavior, on the other hand, where the SPA practically does not increase under coupling, has not to our knowledge been reported before.

In both cases Fig. 2 and Fig. 3 clearly demonstrate that at fixed coupling parameter J stochastic resonance occurs: The SPA has a maximum over temperature T that, unfortunately, cannot be established analytically. Figure 2 (bottom) shows that while ferromagnetic coupling improves the SPA at any fixed temperature T, this improvement is lost if the coupling becomes too strong. There is thus an optimal coupling strength $J_{\text{opt}}(T)$ for every given temperature, which can even be exactly calculated at any $\omega > 0$. One finds

$$J_{\text{opt}}(T) = \frac{T}{2}\,\text{arctanh}\left(1 + \frac{1}{2}\frac{\omega^2}{\alpha^2} - \frac{1}{2}\frac{\omega}{\alpha}\sqrt{4+\frac{\omega^2}{\alpha^2}}\right), \tag{31}$$

which is always positive. Since the SPA does not have a further extremum over J, it implies that antiferromagnetic coupling always decreases the SPA. Moreover, $J_{\text{opt}}(T)$ increases with growing temperature T and decreases as the signal frequency grows. In Fig. 2, the SPA at the optimal coupling strength is included (dash-dotted curves). In Fig. 3 (bottom), where the high-frequency SPA is illustrated, the coupling-induced improvement of the SPA is hardly detectable and $J_{\text{opt}}(T)$ is almost zero.

Considering the SPA of time-independent signals ($\omega=0$) given by Eq. (24), one finds that it has a maximum over temperature for antiferromagnetic coupling $J<0$ only, whereby it decreases as the coupling strength $|J|$ grows. For $J\geqslant 0$ the SPA increases with increasing J as well as with decreasing T. It formally diverges for $T\to 0$ and $J\to\infty$, respectively. In both cases the weak-signal limit (26) breaks down.

In general, the SPA (23) is given by its static value (24) times a dynamical factor $(1+d^2)^{-1}$ [cf. Eq. (16)]. Here d is the ratio $d=\omega/[\alpha(1-\gamma)]$ of signal frequency to long-time relaxation rate, which also governs the phase shift ψ in Eq. (16). With growing d the elements gradually lose their ability to follow the signal: The SPA weakens and the phase shift grows. This effect occurs, for example, if the signal frequency ω increases. Subsequently, the SPA decreases with growing ω.

The impact of d is also responsible for the occurrence of stochastic resonance and optimal coupling in the SPA. Here a decrease of the long-time relaxation rate $\alpha(1-\gamma)$ plays the crucial role: The element's dynamics slows down as J increases or as T decreases. Thus this slow-down occurs whenever the static SPA grows. Hence, the dynamical factor always counteracts the static SPA as T or J is changed. Eventually, the increase of the monotonous static SPA is outperformed by the decrease of the dynamical factor, which results in a maximum of the SPA over T and J, respectively. In other words, stochastic resonance and optimal coupling occur.

As illustrated in Fig. 2, the SPA can be seen as a transition between two limits, the static SPA given by $\rho_s=(q_s/H_0)^2$ ($d\ll 1$, dashed curves) and ρ_s/d^2 ($d\gg 1$, dotted curves), respectively. Since both limits intersect at $d=1$, i.e., at $\omega=\alpha(1-\gamma)$, the plots nicely show the well-known approximate matching of time scales at the SPA peak. This matching not only occurs over T (Fig. 2, top), but over J (Fig. 2, bottom), too. If the signal frequency is changed, the peaks of the SPA shift, whereby all curves share the static SPA as a limit. This results in the qualitative frequency dependence found by McNamara and Wiesenfeld for the δ-function part of the spectrum of the double-well system (4).

Roughly speaking, an improvement of the SPA under ferromagnetic coupling only occurs at $d<1$, where the impact of the static SPA is not yet outperformed by the dynamical factor. This explains why there is no improvement at signal frequencies $\omega\gg 1$: Since under ferromagnetic coupling $\alpha(1-\gamma)\leqslant 1$ holds, $d<1$ cannot be fulfilled in this case. At $\omega=10$ (Fig. 3), the SPA is already within the linewidth of the limit $\rho_s/d^2=\alpha^2(1-\gamma^2)(\omega T)^{-2}$, where the impact of the dynamical factor prevails. This limit is basically given by the modulation term in Eq. (15). Its nonmonotonous temperature dependence is known to be no longer associated with a matching of time scales [10].

The static response, which is thus the origin of the coupling-induced enhancement of the SPA, can be interpreted as follows. Without coupling and for a given fixed signal, the two-state elements prefer to be in the state of lowest energy (8). Since the signal is homogeneous, this state is the same for all elements. Without a signal but under ferromagnetic coupling, neighboring elements tend to be in the same state, too, although they do not favor a particular state $+1$ or -1. Taking now the signal and coupling together, both effects add up. The tendency to find the elements in the state favored by the signal grows and hence q_s and the static SPA grow compared to the uncoupled case.

With antiferromagnetic coupling, on the other hand, neighboring elements prefer to be in opposite states. This counteracts the effect of the signal and leads to a decrease in the SPA.

Finally, the decrease of the static SPA with growing temperature directly follows from Eq. (9). There the imbalance in the distribution of probability between the two states, which depending on the view taken either differ in energy or barrier height, decreases as the system heats up: The mean response q_s weakens due to increasing fluctuations.

V. COUPLING AND SIGNAL-TO-NOISE RATIO

The investigation of the SNR presented here basically relies on numerical evaluations of Eq. (25). An explicit calculation of the power spectrum (21) already yields a rather complicated expression that does not lend itself to a detailed analytical study.

At finite signal frequencies $\omega>0$ the SNR displays the same qualitative dependence on temperature T and coupling parameter J as shown for the SPA in Fig. 2 and Fig. 3, respectively. At low frequencies the SNR is enhanced under ferromagnetic coupling, whereas at high frequencies it is not. Due to this close similarity we omitted the respective plots for the SNR. We note, however, that this similarity is by no means a trivial result, since the spectrum (21) is itself a nonmonotonous function of T and J.

The low-frequency SNR qualitatively reproduces the SNR behavior found in a chain of next-neighbor-coupled overdamped double-well systems simulated in [8]. To our knowledge, it is the first analytical confirmation of this behavior. We note that this correspondence occurs although the simulation in [8] was performed with strong forcing while the present model is studied within a weak-signal limit. As for the SPA, the high-frequency behavior of the SNR has to our knowledge not been reported before.

If the elements are not coupled, their SNR (3) does not depend on the signal frequency. This was already found in [10], if the signal-induced reduction of the continuous part of the spectrum is neglected, as Eq. (3) clearly shows. Under coupling this frequency independence of the SNR is lost. For ferromagnetic coupling, numerical evaluations of Eq. (25) predict a decrease of the SNR with growing signal frequency. Only in the limit of low and high frequencies does the SNR approach constant values. In both cases the SNR formula (25) simplifies significantly. One finds

$$R_s=R_0(1+\gamma)^2=R_0\left[\tanh\left(\frac{2J}{T}\right)+1\right]^2, \tag{32}$$

$$R_{\mathrm{HF}}=R_0\sqrt{1-\gamma^2}=R_0\,\cosh^{-1}\left(\frac{2J}{T}\right), \tag{33}$$

where R_0 is the simplified McNamara-Wiesenfeld SNR (3) of uncoupled elements. R_s is the static SNR while R_{HF} rep-

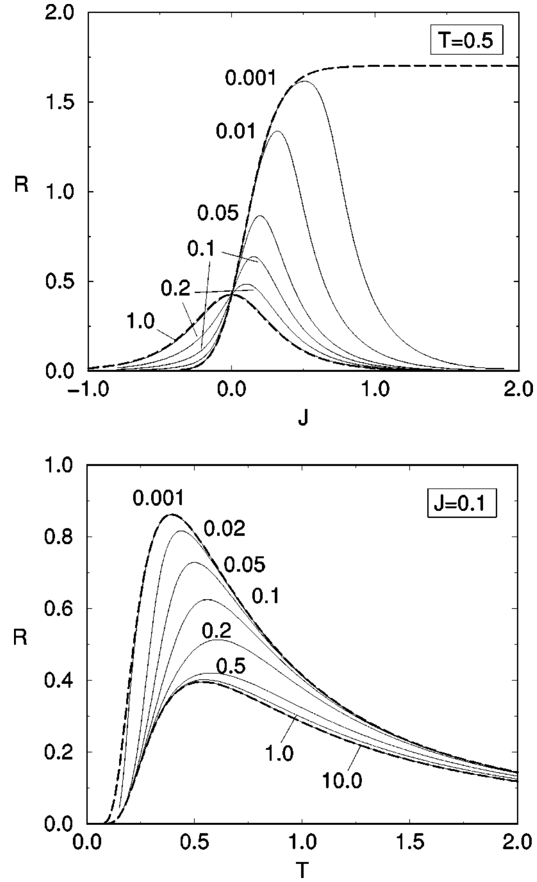

FIG. 4. The frequency dependence of the SNR R. The upper dashed curves represent the static SNR. The lower dashed curves show the high-frequency expansion of the SNR.

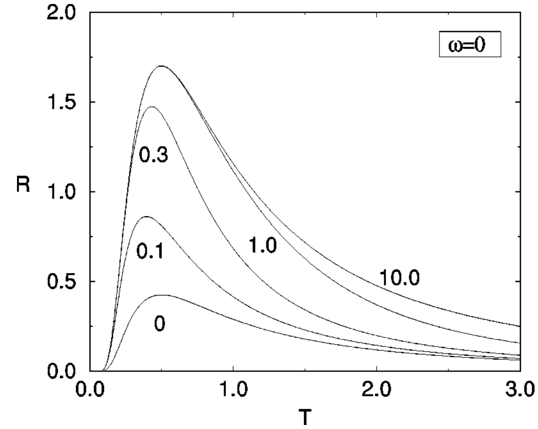

FIG. 5. Static SNR R for various values of the coupling parameter J. The SNR at $J=10$ is within the linewidth of the amplification limit $4R_0$.

resents the leading-order term of its high-frequency expansion. Figure 4 illustrates the typical qualitative behavior of the SNR as a function of the signal frequency. At various frequencies it shows a set of SNR curves over temperature T and coupling parameter J, respectively. The dashed curves represent R_s and R_{HF}, respectively.

The apparent fact that the SNR decreases with growing signal frequency implies that its static value cannot be exceeded at any other signal frequency. Then it follows immediately from Eq. (32) that the coupling-induced enhancement of the SNR possesses an upper limit: It cannot be better than a factor 4. This is a rigorous result for the general chain model and not limited to the prototype system, where a special choice of the parameters was made. Due to the simplicity of the chain model this result may well reflect a limit of a more general nature for the improvement of a weak-signal SNR. Figure 5 shows the static SNR over temperature T. It differs qualitatively from the static SPA, which according to Eq. (24) has a monotonous behavior.

Turning briefly to antiferromagnetic coupling, one finds that the continuous part of the spectrum is insensitive to the sign of the coupling parameter J. Now even the static SNR decreases under coupling. As shown in Fig. 4 (bottom), it is found below the high-frequency SNR curve, i.e., here the SNR increases with growing signal frequency. Since at medium frequencies again a transition occurs between both curves similar to the transition in Fig. 4 (top), a second maximum in the SNR over T may emerge in this situation.

The investigation of the impact of the coupling on the SNR can also be extended to the more natural situation, where the signal itself is embedded into noise. The question to address is whether the improvement of the SNR, which was established for independent internal noise sources, can still be found with additional coherent external noise. (This problem does not occur for the SPA, where only the height of the signal peak is of interest.)

To study this case, it is assumed that the input spectrum consists of the previous signal peak described by $\pi H_0^2\delta(\Omega-\omega)$ and a noise part $N(\Omega)$. Within the weak-signal limit it was shown that the signal peak of the input spectrum leads to a respective peak in the output spectrum at signal frequency only: No additional peaks at multiples of that frequency occur. Therefore, any additional contribution to the continuous part of the output spectrum at signal frequency can only arise from $N(\Omega=\omega)$. In analogy to Eq. (16), one finds that this additional contribution is given by $(q/H_0)^2N(\omega)$.

Together with Eq. (25), the resulting SNR R_{noisy} reads, again in units of H_0^2,

$$R_{\mathrm{noisy}}=\frac{\pi q^2}{H_0^2S(\omega)+q^2N(\omega)}. \tag{34}$$

It can be expressed in terms of the SNR R [cf. Eq. (25)] and the input SNR $R_{\mathrm{input}}=\pi/N(\omega)$ written in units of H_0^2, respectively. One finds

$$\frac{1}{R_{\mathrm{noisy}}}=\frac{1}{R}+\frac{1}{R_{\mathrm{input}}}. \tag{35}$$

R_{noisy} is thus a steadily growing function of the previously studied SNR R, whereby it cannot exceed the input SNR R_{input}. (The latter meanwhile is a well-known result of linear response theory [18], which is in fact the limit we are taking here.) Since R_{input} is constant at fixed signal frequency ω, a coupling-induced increase of R will lead to an increase in R_{noisy}, too. The improvement of the SNR under coupling is thus preserved with external coherent noise. From Eq. (35) one can easily show that the maximum enhancement of

R_{noisy} compared to the uncoupled chain reaches the previously found factor 4 for vanishing input noise only. In general, this factor is smaller approaching 1 if the input noise is so strong that the input SNR and hence R_{noisy} go to zero.

At the end we shall make a brief excursion to the collective response of the chain, which was mentioned in the Introduction. We consider the following sum, which involves the states of N elements:

$$M(t)=\sum_{i}^{N} \sigma_i(t). \tag{36}$$

It is not difficult to show that the SPA of this new quantity is simply N^2 times the previously studied SPA (23). For the SNR, however, this new situation is completely different. Previously the continuous part of the spectrum was determined from the autocorrelation function $c_{kk}(t,\tau)$ of an element alone [cf. Eq. (18)]. Now this spectrum will involve cross-correlation contributions $c_{jk}(t,\tau)$ of different elements $j\neq k$, too.

The new contributions to the continuous part of the spectrum change the qualitative behavior of the SNR under coupling. We will demonstrate this in the limit $N\to\infty$. For this case Glauber calculated the unperturbed spectrum [11], which reads in its one-sided version

$$S_M(\Omega)=N\frac{4\alpha\sqrt{1-\gamma^2}}{\alpha^2(1-\gamma)^2+\Omega^2}. \tag{37}$$

Inserting this spectrum into Eq. (25), multiplied by N^2 due to the mentioned increase of the SPA, one finds for the new SNR per element

$$R_M=\frac{\pi}{4}\alpha\sqrt{1-\gamma^2}\,\delta^2, \tag{38}$$

which, of course, again reduces to the simplified McNamara-Wiesenfeld SNR (3) for vanishing coupling.

Clearly, the SNR is now a decreasing function of the coupling parameter and the sign of the latter is no longer important. Hence, there is a drastic difference in the SNR of single and collective response with respect to coupling. Of course, we have so far merely investigated the limits of Eq. (36), $N=1$ and $N\to\infty$, respectively. We expect that the SNR of the collective response of only a few elements still increases under coupling [16]. A detailed analysis of the present model with respect to these collective effects will be the subject of a further investigation.

VI. SUMMARY

In this paper we extended the two-state model of stochastic resonance introduced by McNamara and Wiesenfeld to a chain of infinitely many coupled two-state elements, which are periodically modulated. The interaction of the elements was chosen in such a way that the chain evolves according to Glauber's stochastic Ising model. In analogy to the work of McNamara and Wiesenfeld and based on Glauber's results, analytical expressions for the signal-to-noise ratio (SNR) and the spectral power amplification (SPA) have been obtained in the limit of weak modulations. Here both quantities refer to the response of a single element as part of the chain.

Instead of approximating the dynamics of a particular coupled system by the chain model, we used the latter to build a prototype system that hopefully captures the essential features of an entire class of coupled stochastic resonators. To this end additional assumptions were made on the dependence of the model parameters on some noise intensity and signal amplitude, respectively. The prototype system was used to study the effect of the coupling on the response of a single resonator embedded into the chain.

The results show that array-enhanced stochastic resonance occurs for ferromagnetic-type coupling in SPA and SNR. The qualitative features of the effects reproduce those previously found in coupled stochastic resonators. The simplicity of the chain model allowed for a detailed analytical investigation of the SPA. For the SNR the observed effects have been confirmed analytically. In addition, it was found that the improvement of the single-resonator response compared to the response of the uncoupled resonator still occurs, if the signal is embedded into noise. For the SNR this improvement was shown to be limited by a factor 4, which is reached for vanishing input noise only. A brief excursion into the collective response of N resonators, on the other hand, disclosed that coupling cannot improve the SNR, if N is very large.

A closer look at the mechanisms behind the effects revealed that in the present model an improvement of the stochastic resonators under coupling is associated with the regime of quasistatic response. Since the model studied here possesses an upper bound to the time scale of its dynamics, the desired improvement is essentially restricted to sufficiently slow signals. For the SPA this improvement is based on a stronger tendency of the two-state elements to align in parallel, if signal and ferromagnetic coupling act together, compared to this tendency caused by the signal alone. The reason why the improvement of the SPA is lost, if the coupling is too strong, was found to be the slow-down of the system dynamics under coupling: It simply prevents the resonators from responding quasistatically.

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